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Some covariant representations of massless boson fields

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Abstract. Consider the c^*-algebra of the canonical commutation relations (CCR), acted on by a group G of one-particle symmetry transformations V. A symplectic operator T defines a representation $\pi_T = \pi_F \circ T$ where π_F is the Fock representation. The automorphisms of the CCR algebra that are induced by G are shown to be continuously implemented in π_T if and only if $A - V(g)AV^*(g)$ is a continuous Hilbert–Schmidt 1-cocycle of G; here, A is related to T by $T = \exp A$, A being a suitable bounded, anti-linear self adjoint operator.

Some new examples of fully Poincaré-covariant representations of massless fields in $1+1$ dimensions are constructed.

Introduction

This paper continues in the spirit of previous works (Streater and Wilde 1970, Streater 1971, Roepstorff 1970, Bonnard and Streater 1975, 1976, Basarab-Horwath *et al* 1979, Kraus *et al* 1977a, b, Kraus and Streater 1981), concerned with finding representations of the free quantised field that are covariant under some group, have positive energy and which are inequivalent to the Fock representation. Such representations are interesting in the description of states with an infrared problem (Kraus *et al* 1977a, Reents 1977) and in some exactly soluble models (Streater 1974a, b).

There are several interesting c^*-algebras associated with the free boson field, for example the local algebra of Haag and Kastler (1964), and the minimal algebra of Manuceau (1968) and Slawny (1972). For definiteness, we use the original algebra defined by Segal (1959) for which the results of Shale (1962) apply. In this section we summarise the concepts we shall need. In § 2 we derive two results that are new, and in § 3 we give the detailed construction of the new covariant representations of the free boson field in $1+1$ dimensions.

Let $\mathscr{H}$ be a complex Hilbert space. The CCR algebra $\mathscr{A}$ over $\mathscr{H}$ is the set of finite formal sums (over $\mathbb{C}$) of elements of $\mathscr{H}$: thus $\Sigma_1^n \alpha_i[h_i] \in \mathscr{A}$, $\alpha_i \in \mathbb{C}$, where $[h_i]$ is a formal element of $\mathscr{H}$. The multiplication law is determined by

$$[h_1][h_2] = \exp(\tfrac{1}{2}\mathrm{i}\,\mathrm{Im}\langle h_1, h_2\rangle)[h_1 + h_2], \qquad (1)$$

A representation of the CCR algebra is a pair $(W, \mathscr{K})$ where $\mathscr{K}$ is a complex Hilbert space and W is a map $W: \mathscr{H} \to \mathrm{Aut}\,\mathscr{K}$ (the set of unitaries on $\mathscr{K}$) obeying

$$W(h)W(h') = \exp(\tfrac{1}{2}\mathrm{i}\,\mathrm{Im}\langle h, h'\rangle)W(h+h') \qquad (2)$$

and such that, for each finite-dimensional subspace $\mathscr{H}_0 \subseteq \mathscr{H}$, the map $W: h \to W(h)$ is strongly continuous as a map $\mathscr{H}_0 \to \mathrm{Aut}\,\mathscr{K}$. The Stone–von Neumann theorem asserts that all irreducible representations of the CCR over $\mathcal{M}$, where $\dim \mathcal{M} < \infty$, are equivalent. As a result, the two W^*-algebras $\mathfrak{A}_1(\mathcal{M})$, $\mathfrak{A}_2(\mathcal{M})$ generated by two representations W_1, W_2 of the CCR over $\mathcal{M}$, are algebraically isomorphic under the map which sends $W_1(h)$ to $W_2(h)$. Thus, the set of W^*-algebras $\{\mathfrak{A}(\mathcal{M})\}$ as $\mathcal{M}$ runs over the finite-dimensional subspaces of $\mathcal{H}$, forms an inductive system. The inductive limit is the c^*-algebra $\mathfrak{A}(\mathcal{H})$ of Segal (1959).

A real-linear operator T on $\mathcal{H}$ (ie., $\alpha T = T\alpha$ for real α) is called symplectic if

$$\operatorname{Im}\langle Tf, Tg\rangle = \operatorname{Im}\langle f, g\rangle \tag{3}$$

for all $f, g \in \mathcal{H}$. We shall consider the set of bounded invertible symplectic operators; these form a group. A unitary operator is clearly symplectic.

A symplectic operator T defines an automorphism σ_T of $\mathfrak{A}(\mathcal{H})$ namely, the unique automorphism such that $\sigma_T(W(h)) = W(Th)$. Then a group $\{\sigma_\alpha : \alpha \in G\}$ of automorphisms is obtained from a group $\{T_\alpha : \alpha \in G\}$ of symplectics; in particular for the free relativistic quantised field, $\mathcal{H}$ carries a representation $\{V(L);\ L = (a, \Lambda) \in \mathcal{P}_+^\uparrow\}$ of the Poincaré group, leading to the automorphism group $\{\sigma_L : L \in \mathcal{P}_+^\uparrow\}$.

A quasi-free state ω is determined by

$$\omega(W(h)) = e^{i \operatorname{Im}\langle \Phi^\times, h\rangle}\, e^{-\frac{1}{4}\langle h, Bh\rangle} \tag{4}$$

for some operator B on $\mathcal{H}$ and some $\Phi^\times \in \mathcal{H}^\times$, the algebraic dual to $\mathcal{H}$. The case $B = 1$, $\Phi^\times \neq 0$ has been treated before (Basarab–Horwath *et al* 1979); the case $B = 1$, $\Phi^\times = 0$ is the Fock vacuum state, ω_F. Shale (1962) considers the class of states ω with $\Phi^\times = 0$ that are determined by a symplectic operator T with

$$\omega(W(h)) = \omega_F(W(Th)) \tag{5}$$

We write $\omega_F \circ T$ or ω_T for ω in this case. The corresponding representation will be written π_T:

$$\pi_T(W(f)) = W_F(Tf). \tag{6}$$

Let us say $T_1 \sim T_2$ if π_{T_1} and π_{T_2} are equivalent. It is known that π_F and π_T are equivalent if and only if (Shale 1962):

$$1 - |T| = 1 - (T^\dagger T)^{1/2} \tag{7}$$

is a Hilbert–Schmidt operator on $\mathcal{H}$ regarded as a real vector space and $\dagger$ is the adjoint relative to $\operatorname{Re}\langle\, ,\rangle$. Our first task is to obtain a more convenient 'linear' condition equivalent to (7).

2. A linear implementability condition

A real-linear map T on $\mathcal{H}$ can be uniquely expressed as

$$T = T_+ + T_- \tag{8}$$

where T_+ is $\mathbb{C}$-linear and T_- is anti-linear. Then T is symplectic iff

$$T^{-1} = T_+^* - T_-^* \tag{9}$$

where the adjoint of T_- is defined by

$$\langle f, T_- g\rangle = \overline{\langle T_-^* f, g\rangle}. \tag{10}$$

Let $T_+ = U|T_+|$ be the polar decomposition of T_+. Since the automorphism σ_U is implemented in π_F (by $\Gamma(U)$), $T \sim |T_+| + U^{-1}T_-$. Thus, each equivalence class contains a symplectic with positive linear part. Let T be chosen so that T_+ is positive; we now show that T_- is then self-adjoint. From (9)

$$T_+^*T_+ - T_-^*T_- = 1 \tag{11a}$$

$$T_+T_+^* - T_-T_-^* = 1 \tag{11b}$$

$$T_+^*T_- - T_-^*T_+ = 0 \tag{11c}$$

$$T_+T_-^* - T_-T_+^* = 0. \tag{11d}$$

If $T_+ = T_+^*$, subtract the first two lines: T_- is normal. Multiply (11a, b) fore and aft by T_-; subtract, to get $[T_+^2, T_-] = 0$, so $[T_+, T_-] = 0$. From (11c, d) we get $T_+[T_- - T_-^*] = 0$, so $T_- = T_-^*$ as $T_+ \geq 1$.

We can write T_- as the product of a positive linear self-adjoint operator β_- and the conjugation C defined on $(\mathrm{Ker}\ T_-)^\perp$ by the closure of

$$C_1 = T_-(T_-^2)^{-1/2} \tag{12}.$$

C_1 commutes with $\beta_- := \bar{C}_1 T_-$ and with $\beta_+ := T_+$ (which is a function of β_-, by (11a)). Let C_2 be any conjugation on Ker T_-. Then $C = \bar{C}_1 \oplus C_2$ commutes with $\beta_\pm$; and $T_- = C\beta_-$. It follows that $\beta_- = CT_- = (T_-^2)^{1/2}$. So, $T = \beta_+ + \beta_-$ on $\mathrm{Re}\ \mathscr{H} = \{f: Cf = f\}$, and $T = \beta_+ - \beta_-$ on $\mathrm{Im}\ \mathscr{H} = \{f: Cf = -f\}$. Then, by (9), $(\beta_+ + \beta_-)^{-1} = \beta_+ - \beta_-$, so T takes the standard form $S \oplus S^{-1}$: $S = (\beta_+ + \beta_-)\restriction \mathrm{Re}\ \mathscr{H}$.

It is known (Shale 1962, Berezin 1966) that $T \sim 1$ if and only if $T_- \in B(\mathscr{H})_2^c$ (the set of anti-linear Hilbert–Schmidt operators on $\mathscr{H}$); the Hilbert–Schmidt norm for anti-linear operators (as well as for linear operators) h being defined by

$$\|h\|_2^2 = \sum_k \|hf_k\|^2 = \sum_{i,k} |\langle g_i, hf_k\rangle|^2 \tag{13}$$

where $\{f_k\}$ and $\{g_i\}$ are orthonormal bases.

Let $U(T)$ implement σ_T when $T \sim 1$. The phase of $U(T)$ is fixed by $\langle \Omega, U(T)\Omega\rangle > 0$, where Ω is the Fock vacuum. Let $\mathscr{F}$ denote the Fock space over $\mathscr{H}$. It is known (Shale 1962) that:

$$U(T) \to 1 \text{ strongly in } \mathscr{F} \tag{14}$$

if

$$T_+ \to 1 \text{ strongly in } \mathscr{H} \tag{15a}$$

and

$$T_- \to 0 \text{ in } B(\mathscr{H})_2^c \tag{15b}$$

In (15a), T_+ is not necessarily in standard form. We prove this and the converse, which seems to be new.

Let $K_n(x_1, \ldots, x_n)$ be the n-particle contribution to $U(T)\Omega$. Then (Berezin 1966, theorem 4.1)

$$\begin{aligned} K_{n+1}(x, x_1, \ldots, x_n) &= -[n(n+1)]^{-1/2}\{A(x, x_1)K_{n-1}(x_2, \ldots, x_n) + \cdots \\ &\quad + A(x, x_n)K_{n-1}(x_1, \ldots, x_{n-1})\}. \end{aligned} \tag{16}$$

$A(x, y)$ being the kernel of the $B(\mathscr{H})_2^c$ operator $T_+^{-1}T_-$ (in Berezin's notation, $A = \Phi^{-1}\Psi$). In particular, the norm of the two-particle contribution is given by

$$\|K_2\|^2 = \tfrac{1}{2}\|A\|_2^2|K_0|^2 \tag{17}$$

and the norm of the $2n$-particle contribution can be estimated by

$$\|K_{2n}\|^2 \leq \|A\|_2^{2n}|K_0|^2 \tag{18}$$

so that the norm of the part of $U(T)\Omega$ orthogonal to Ω is less than

$$[(1-\|A\|_2^2)^{-1}-1]|K_0^2|. \tag{19}$$

Now, $\|A\|_2 = \|T_+^{-1}T_-\|_2 \leq \|T_+^{-1}\|\|T_-\|_2 \leq \|T_-\|_2 \to 0$ as $\|T_-\|_2 \to 0$; also $|K_0| \leq 1$. Hence $(19) \to 0$ and $U(T)\Omega \to \Omega$ as $\|T_-\|_2 \to 0$. Now consider the total set $\{W_{\rm F}(g)\Omega \colon g \in \mathscr{H}\} \subseteq \mathscr{F}$. By definition of $U(T)$

$$U(T)W_{\rm F}(g)\Omega = W_{\rm F}(Tg)U(T)\Omega.$$

Since $T \to 1$ strongly in $\mathscr{H}$ (equation (15)), $W_{\rm F}(Tg) \underset{s}{\to} W_{\rm F}(g)$ (see Araki and Woods 1963). Hence, if (15) holds, $W_{\rm F}(Tg)U(T)\Omega \to W_F(g)U(T)\Omega$ and $U(T) \to 1$ strongly, as all operators are unitary and so are uniformly bounded in norm. This shows that $(15) \Rightarrow (14)$.

Conversely, (14) implies $U(T)\Omega \to \Omega$ so that $(17) \to 0$ and $\|A\|_2 = \|T_+^{-1}T_-\|_2 \to 0$. By Berezin's identity (4.19)

$$(T_+^*T_+)^{-1} = 1-(T_+^{-1}T_-)(T_+^{-1}T_-)$$

we see that $\|T_+\| \leq 1+\varepsilon$ if $\|T_+^{-1}T_-\|$ ($\leq \|T_+^{-1}T_-\|_2$) is sufficiently small. Therefore, $\|T_-\|_2 \leq \|T_+\|\|A\|_2 \to 0$ proving (15b). Finally,

$$W_{\rm F}(Tg) = U(T)W_{\rm F}(g)U^*(T) \underset{s}{\to} W_{\rm F})$$

as $U(T) \to 1$ strongly and according to Araki and Woods (1963), this implies $T \underset{s}{\to} 1$, proving (15a).

Another standard form of symplectic operators is determined by a bounded linear self-adjoint operator $\alpha = \ln(\beta_+ + \beta_-)$. This makes sense because $\beta_+ = (1+\beta_-^2)^{1/2}$, so $\beta_+ \pm \beta_- > 0$. Since C commutes with β_+ and β_- we have $[\alpha, C] = 0$. On Re $\mathscr{H}$, $T = \beta_+ + \beta_- = \exp\alpha = \exp\alpha C$; on Im $\mathscr{H}$, $T = \beta_+ - \beta_- = (\beta_+ + \beta_-)^{-1} = \exp(-\alpha) = \exp\alpha C$. Combining, we get

$$T = \exp(\alpha C)$$

$$T_+ = \cosh A = \cosh\alpha, \quad A = \alpha C$$

$$T_- = \sinh A = C\sinh\alpha$$

Thus, every T with $T_+ > 0$ is the exponential of a self-adjoint anti-linear operator. Conversely, an anti-linear self-adjoint operator A defines a symplectic of the standard form $S \oplus S^{-1}$ via $T = \exp A$.

Theorem 1. Let A, B be bounded, anti-linear and self-adjoint. Then $T_A = \exp A$ and $T_B = \exp B$ are equivalent if and only if $\|A - B\|_2 < \infty$.

Proof. To prove the 'if' part, write $B = A + h$, $\|h\|_2 < \infty$, and $\Delta = (\exp(A+h)\exp(-A))_-$. Then we have to show that $\|\Delta\|_2 < \infty$, with

$$\Delta = \sinh(A+h)\cosh A - \cosh(A+h)\sinh A. \tag{20}$$

In the power series expansion of (20) all terms not containing a factor h cancel (since $[\cosh A, \sinh A] = 0$) and for the remaining terms the triangle inequality of $B(\mathscr{H})_2^c$ together with $\|AhB\|_2 \leq \|A\|\,\|h\|_2\,\|B\|$ leads to

$$\|\Delta\|_2 \leq \|h\|_2\left[\frac{\cosh(\|A\|+\|h\|)-\cosh\|A\|}{\|h\|}\sinh\|A\| + \frac{\sinh(\|A\|+\|h\|)-\sin\|A\|}{\|h\|}\cosh\|A\|\right]. \tag{21}$$

For the 'only if' part, we need:

Lemma. Any symplectic T is equivalent to some T_p with pure point spectrum.

Proof of lemma. The Weyl–von Neumann theorem (see e.g. Kato 1966) states that for each self-adjoint linear operator A in a complex Hilbert space, a Hilbert–Schmidt operator h can be constructed, such that $A_p = A + h$ is self-adjoint with pure point spectrum. We intend to write $T = \exp(C\alpha)$, $T_p = \exp(C\alpha_p)$ with $\alpha_p = \alpha + h$ and to apply the 'if' part of the theorem already proved, but we get into trouble if $[h, C] \neq 0$. However, inspection of Section X, §2.1 of Kato (1966) shows that h can indeed be chosen so that it commutes with C. Namely, because of $[\alpha, C] = 0$, the spectral projections of α are real operators i.e. $[E_\lambda, C] = 0$; now restrict lemma 2.2 of Kato (1966) to real $f \in \mathscr{H}$, and the proof of theorem 2.1 to Re $\mathscr{H}$; finally, extend the constructed projections complex-linearly to all of $\mathscr{H}$. This proves the lemma.

Proof of the 'only if' part of theorem 1

Let $\exp A$ and $\exp B$ be equivalent symplectic operators. According to the lemma, $A_p = A + h_A$, $B_p = B + h_B$, where A_p and B_p have a pure point spectrum, and $\exp A_p$ is equivalent to $\exp A$, and $\exp B_p \sim \exp B$. Then $\exp A_p \sim \exp B_p$ too, so by Shale's criterion, the anti-linear part of $\exp(A_p)\exp(-B_p)$, namely

$$\sinh A_p \cosh B_p = \cosh A_p \sinh B_p \tag{22}$$

must be a Hilbert–Schmidt operator. We show that $\|A_p - B_p\|_2 < \infty$ from which $\|A - B\|_2 < \infty$ follows.

Let C_A and C_B be the conjugations associated with A_p and B_p respectively obeying (12). Let $\{f_i\}$ be a real eigenbase for A_p, $\{g_k\}$ a real eigenbase for B_p:

$$\begin{aligned} A_p f_i &= \lambda_i f_i, \qquad & C_A f_i &= f_i \\ B_p g_k &= \mu_k g_k, \qquad & C_B g_k &= g_k \end{aligned} \tag{23}$$

We write the $B(\mathscr{H})_2^c$ norm of (22) using (13) with $\{f_i\}$ on the left side, $\{g_k\}$ on the right side, and (10) for the anti-linear operators $\sinh A_p$, $\sinh B_p$. We get

$$\infty > \|(\exp A_p \exp(-B_p))_-\|_2^2$$
$$= \sum_{i,k} |\cosh \lambda_i \langle f_i, g_k\rangle \sinh \mu_k \sinh \lambda_i \overline{\langle f_i, g_k\rangle} \cosh \mu_k|^2$$
$$= \sum_{i,k} |\sinh(\lambda_i - \mu_k) \operatorname{Re}\langle f_i, g_k\rangle|^2 + \sum_{i,k} |\sinh(\lambda_i + \mu_k) \operatorname{Im}\langle f_i, g_k\rangle|^2$$
$$\geqslant \sum_{i,k} |(\lambda_i - \mu_k) \operatorname{Re}\langle f_i, g_k\rangle|^2 + \sum_{ik} |(\lambda_i + \mu_k) \operatorname{Im}\langle f_i, g_k\rangle|^2$$
$$= \|A_p - B_p\|_2^2 \tag{24}$$

This completes the proof of theorem. The 'only if' part of theorem 1 is also valid for unbounded symplectic operators, whereas the 'if' part is not.

We now turn to the question of how convergence of symplectic operators $T = \exp A$ in the sense of (15) is transferred to A.

Theorem 2. A sequence of symplectic operators $T_i = \exp A_i$ converges to $T = \exp A$ in the sense of

$$(T_i T^{-1})_+ \underset{s}{\to} 1; \ \|(T_i T^{-1})_-\|_2 \to 0 \tag{25}$$

if and only if

$$\|A_i - A\|_2 \to 0. \tag{26}$$

Proof. Suppose $\|A_i - A\|_2 \to 0$. Then, by (20) and (21), $\|(T_i T^{-1})_-\|_2 \to 0$. Also, $(T_i T^{-1})_+ = \cosh A_i \cosh A - \sinh A_i \sinh A$ converges to 1 even uniformly, since $A_i \to A$ in norm.

For the converse, we want to generalise (24) to

$$\|(\exp A \exp(-B))_-\|_2^2 \geqslant \|A - B\|_2^2. \tag{27}$$

In the Weyl–von Neumann theorem, the Hilbert–Schmidt norms of the operators h_A, h_B, where $A_p = A + h_A$, $B_p = B + h_B$, can be made arbitrarily small (Kato 1966, X, § 2.1), by a suitable choice of A_p, B_p. We write:

$$\exp A \exp(-B) = \{\exp A \exp(-A_p)\}\{\exp(A_p) \exp(-B_p)\}\{\exp B_p \exp(-B)\}.$$

As $\|h_A\|_2 \to 0$ the linear part of the first factor { } converges to 1 in norm, and the anti-linear part goes to zero in $B(\mathscr{H})_2^c$, by (21) (since $\|A_p\| \to \|A\|$, the numbers $\|A_p\|$ needed in (21) remain bounded). Similarly for the last factor {} as $\|h_B\|_2 \to 0$. Hence $\|(\exp A \exp(-B))_-\|_2$ differs from $\delta = \|(\exp A_p \exp(-B_p))_-\|_2$ by a term going to zero with $\|h_A\|_2$, $\|h_B\|_2$; by (24), $\delta \geqslant \|A_p - B_p\|_2$, which in turn converges to $\|A - B\|_2$. This proves (27). The conclusion from (25) to (26) is now obvious.

3. Construction of covariant representations

Suppose now that $\mathscr{H}$ carries a unitary representation V_g of a group G, leading to an automorphism group $\{\tau_g : g \in G\}$ of $\mathfrak{A}(\mathscr{H})$. Let π_T be a representation of $\mathfrak{A}(\mathscr{H})$ of the form $\pi_T(w(f)) = \pi_F(w(Tf))$. The automorphisms $\{\tau_g\}$ are implemented in the representation π_T if there exists a unitary operator U_g for each $g \in G$, such that

$$U_g \pi_T(W(f)) U_g^{-1} = \pi_T(W(V_g f)),$$

that is

$$U_g W_F(Tf) U_g^* = W_F(TV_g f).$$

Hence, τ_g is implemented in π_T if and only if $T \sim TV_g$. If T is in standard form (of § 2) then TV_g is not in general of standard form. But $V_g^* TV_g$ is of standard form, and $V_g^* TV_g \sim TV_g$.

Let $T = \exp A$ where $A = C\alpha$ as in § 2. Then $V_g^* TV_g = \exp(V_g^* A V_g)$. Write

$$A_g = A - V_g^* A V_g. \tag{28}$$

The map $g \mapsto A_g$ obeys the 'cocycle equation' $V_g^* A_h V_g = A_{hg} - A_g$.

By theorem 1, if $T = \exp A$, $T' = \exp A'$, then $T \sim T'$ if and only if $A - A' \equiv h \in B(\mathscr{H})_2^c$. The corresponding cocycles then differ by $h - V_g^* h V_g$, a 'coboundary' i.e. of the form (28) with $A = h \in B(\mathscr{H})_2^c$. Let $Z^1(G, B(\mathscr{H})_2^c)$ denote the set of anti-linear self-adjoint Hilbert–Schmidt cocycles A_g of the form (28), and $B^1(G, B(\mathscr{H})_2^c)$ the corresponding coboundaries.

Theorem 3. Consider an irreducible representation of a simply connected symmetry group G on the 1-particle space $\mathscr{H}$ such that at least one of the generators (e.g. the energy H) is positive. Then there is a one-to-one correspondence between G-covariant quasi-free representations of the CCR of the form π_T, and elements of the anti-linear $B(\mathscr{H})_2^c$-valued cohomology group $H^1(G, B(\mathscr{H})_2^c) = Z^1(G, B(\mathscr{H})_2^c)/B^1(G, B(\mathscr{H})_2^c)$. The projective representation of G in π_T is continuous if and only if the corresponding cocycle is (equivalent to one that is) continuous in $B(\mathscr{H})_2^c$.

Proof. It is proved in the remarks above that a G-covariant representation π_T, $T = \exp A$, defines an element of $H^1(G, B(\mathscr{H})_2^c)$. Now let $A_g = A - V_g^* A V_g$ and $B_g = B - V_g^* B V_g \in Z^1(G, B(\mathscr{H})_2^c)$ such that $A_g - B_g = h - V_g^* h V_g \in B^1(G, B(\mathscr{H})_2^c)$. Then $A - B - h = F$ is self-adjoint, anti-linear and commutes with V_g for all $g \in G$. Note that Schur's Lemma does not hold in general for anti-linear operators, but F^2 is linear (and positive), obeys $[F^2, V_g] = 0$ and thus $F^2 = \lambda 1$, $\lambda \geq 0$. Assume $\lambda \neq 0$; then $C = F\lambda^{-1/2}$ is a conjugation and $[C, V_g] = 0$. Let $V_{g(t)} = \exp(\mathrm{i}Ht)$ with $H \geq 0$ and $f \in D_H$, $Hf \neq 0$. From

$$\mathrm{i}Hf = \lim_{t \to 0} t^{-1}(V_{g(t)} - 1)f$$

we conclude $HCf = -CHf$, thus $Cf \in D_H$ and $\langle Cf, HCf \rangle = -\langle f, Hf \rangle < 0$ which contradicts the positivity of H. Hence $\lambda = 0$, i.e. $A - B = h$ so that the representations π_{T_A} and π_{T_B} are equivalent.

For the continuity, suppose U_g implements τ_g in π_T. Then

$$U_g W_F(Tf) U_g^* = W_F(TV_g f) \tag{29}$$

so

$$U_g W_F(f) U_g^* = W_F(TV_g T^{-1} f). \tag{30}$$

Thus, U_g implements the automorphism generated by the symplectic operator $TV_g T^{-1}$, in π_F. By Shale's theorem 4.2 and its converse proved in § 2, U_g is continuous at $g = \mathrm{id}_G$

(and hence defines a continuous projective representation everywhere) iff for $g \to \mathrm{id}_G$, $TV_gT^{-1} \to 1$ in the sense of (15); thus $(V_g^*TV_g)T^{-1} \to 1$ in that sense. Write $T = \exp C\alpha = \exp A$. Then $V_g^*TV_g = \exp(V_g^*AV_g)$ and by theorem 2, $V_g^*TV_g \to T$ in the sense of (15) if and only if

$$\|V_g^*AV_g - A\|_2 \to 0 \quad \text{for} \quad g \to \mathrm{id}_G. \tag{31}$$

Thus, U_g is continuous at id_G if and only if A_g is a continuous cocycle at id_G and hence everywhere. This proves the theorem. It is obvious that if A_g is a continuous cocycle, then so is $(A+h)_g$ with $h \in B(\mathscr{H})_2^c$.

To get some new examples of $\mathscr{P}_+^\uparrow$-covariant representations of the free boson field in 1 + 1 dimensions, we construct non-trivial elements of $H^1(\mathscr{P}_+^\uparrow, B(\mathscr{H})_2^c)$ as follows. We take $\mathscr{H} = L^2(\mathbb{R}, \mathrm{d}p/|p|)$ which carries the usual representation V of zero mass with one-particle Hamiltonian h. We choose C to be the complex conjugation in this momentum space. This has the advantage that C commutes with Lorentz transformations (but not with space–time translations).

In Kraus and Streater (1981) functions $\{f_j\}$ were constructed with the following properties: $f_j(p)$ real, and:

$$\text{The supports of } f_j \text{ are mutually disjoint} \tag{32}$$

$$\sum_j \langle f_j, hf_j\rangle < \infty, \qquad \|f_j\| = 1, j = 1, 2, \ldots \tag{33}$$

$$\sum_j \|f_j - V_\Lambda f_j\|^2 \leq B|\lambda|, \qquad B = 2\sum_{j=1}^{\infty} j^{-(1+\varepsilon)} \tag{34}$$

where λ parametrises the Lorentz transformation Λ and is zero for $\Lambda = 1$. The existence of a non-trivial cocycle of $H^1(\mathscr{P}, B(\mathscr{H})_2^c)$ can be proved as in Kraus and Streater (1981). Thus: $T = \exp(\alpha C)$ with C as above and

$$\alpha = \sum_j |f_j\rangle\alpha_j\langle f_j| = \sum |f_{jk}\rangle\alpha_{jk}\langle f_{jk}|$$

where $\{f_{jk}\}$ extends $\{f_j\}$ to a complete real orthonormal system with disjoint supports for different j, and $f_{j0} = f_j$, $\alpha_{jk} = 0$ if $k \neq 0$, $\alpha_{j0} = \alpha_j$. Clearly π_T is non-Fock if

$$\sum_{jk} |\alpha_{jk}|^2 = \sum_j |\alpha_j|^2 = \infty \tag{35}$$

For the Hilbert–Schmidt norm of the cocycle we have

$$\begin{aligned}
\|A_\Lambda\|_2^2 = \|V_\Lambda A_\Lambda\|_2^2 &= \|\alpha C V_\lambda - V_\lambda(\alpha C)\|_2^2 \\
&= \sum_{jkj'k'} |\langle f_{jk}, ((\alpha C)V_\lambda - V_\lambda \alpha C)f_{j'k'}\rangle|^2 \\
&= \sum_{jkj'k'} |\langle f_{jk}, (\alpha V_\lambda - V_\lambda\alpha)f_{j'k'}\rangle|^2 \\
&= \sum_{jkj'k'} |(\alpha_{jk} - \alpha_{j'k'})\langle f_{jk}, V_\lambda f_{j'k'}\rangle|^2
\end{aligned} \tag{36}$$

since $[V_\lambda, C] = 0$ and $Cf_{jk} = f_{jk}$.

In the last term, V_λ can be replaced by $1 - V_\lambda$ since $\langle f_{jk}, f_{j'k'}\rangle = \delta_{jj'}\delta_{kk'}$. Also, for a term of the sum to be non-zero, $k = 0$ or $k' = 0$. Thus

$$(36) \leq \sum_{jj'k'} |(\alpha_{j0} - \alpha_{j'k'})\langle f_{j0}, (1 - V_\lambda)f_{j'k'}\rangle|^2 + \sum_{jj'k} |(\alpha_{jk} - \alpha_{j'0})\langle f_{jk'}(1 - V_\lambda)f_{j'0}\rangle|^2 \tag{37}$$

If $\sup_j |\alpha_j| = s < \infty$, then

$$(36) \leq 4s^2 \sum_j \sum_{j'k'} |\langle (1-V_\lambda^*) f_j, f_{j'k'}\rangle|^2 + 4s^2 \sum_{j'} \sum_{jk} |\langle f_{jk'} (1-V_\lambda) f_{j'} \rangle|^2$$

$$= 8s^2 \sum_j \|f_j - V_\lambda f_j\|^2 \leq 8s^2 B|\lambda|. \quad (38)$$

Thus, (31) is satisfied for Lorentz transformations, which are therefore continuously implemented in π_T.

For space-time translations we can write $\mathscr{H} = \oplus \mathscr{H}_j$, where $\mathscr{H}_j = L^2(M_j, dp/|p|)$, with all M_j disjoint, $\mathbb{R} = M_j$ and $\operatorname{supp} f_j \subseteq M_j$. Then $\mathscr{F}$ is the infinite tensor product $\otimes^\Omega \mathscr{F}(\mathscr{H}_j)$ where Ω is the fiducial vector $\otimes \Omega_j$, Ω_j being the Fock vacuum of $\mathscr{F}(\mathscr{H}_j)$†. We may assume that $\{f_{jk}\}_{k=0,1,\ldots}$ form an orthonormal basis in $\mathscr{H}_j$. Then the state $\omega_T = \omega_F \circ T$ is an infinite product state $\otimes_j \Omega_j \circ T_j$, where T_j is $T \upharpoonright \mathscr{H}_j$. T_j alters just one mode in $\mathscr{H}_j$, namely, the mode f_{j0}. Hence $\Omega_j \circ T_j$ is a Fock state in $\mathscr{F}(\mathscr{H}_j)$. A calculation of the total energy in the state $\otimes_j \Omega_j \circ T_j$ gives $\Sigma_j \sinh^2 \alpha_j \langle f_j, hf_j \rangle$ which is finite by (33) and (37). We can therefore apply the theorem of Kraus *et al* (1976b) to conclude that the infinite product of space–time translation operators on $\mathscr{F} = \otimes \Omega \circ T\mathscr{F}_j$ defines a continuous unitary representation of $\mathbb{R}^2$ obeying the spectral condition. By theorem 3, $\{A_g : g \in \mathbb{R}^2\}$ is a continuous cocycle, and defines a covariant representation of the CCR with positive energy.

A construction similar to that in Kraus and Streater (1981) would construct models in $1+3$ dimensions with positive energy, and covariant under boosts in one direction and rotations about this direction. Representations covariant under $\mathbb{R}^4 \times SO(3)$ can also easily be given. Such representations have already been noted (Kraus *et al* 1977a).

Considerations similar in spirit have been discussed by Carey and Hurst (1979) and Ekhaguere (1978). Basarab-Horwath and Polley (1981) have proved that $H^1(\mathscr{P}, B(\mathscr{H})_2^c)$ is trivial in $(3+1)$ dimensions, and also in $(2+1)$ dimensions.

It is likely that $H^1(\mathscr{P}, B(\mathscr{H})_2^c)$ is trivial in $1+s$ dimensions, for all $s > 1$. The method used here to construct cocycles will not work, as it entails the construction of functions $\{f_j\}$ such that $\Sigma(f_j - V_\lambda f_j)$ is a vector-valued cocycle. Such a construction leads only to coboundaries in $1+s$ dimensions, since $H^1(\mathscr{P}, \mathscr{H}, V)$ is trivial in that case (Basarab-Horwath *et al* 1979, theorem 4).

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† $\mathscr{F}$ denotes the Fock space over $\mathscr{H}$ and $\mathscr{F}(\mathscr{H}_j)$ the Fock space over a subspace $H_j \subseteq H$.

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